Model of the gravitational dipole

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Abstract

A model of the gravitational dipole is proposed in a close analogy to that of the global monopole. The physical properties and the range of validity of the model are examined as is the motion of test particles in the dipole background. It is found that the metric of the gravitational dipole describes a curved space-time, so one would expect it to have a more pronounced effect on the motion of the test particles than the spinning cosmic string. It is indeed so and in the generic case the impact of repulsive centrifugal force results in a motion whose orbits when projected on the equatorial plane represent unfolding spirals or hyperbolas. Only in one special case these projections are straight lines, pretty much in a manner observed in the field of the spinning cosmic string. Even if open, the orbits are nevertheless bounded in the angular coordinate θ.

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1 Introduction

Recently, there has been a flurry of activity in studying exotic cosmic structures, ranging from strings to monopoles to walls [1-4]. The motivation to study these objects comes mainly from theories of particle physics, where such structures naturally emerge as topological defects which may have arisen during phase transitions in the early Universe. Since they are macroscopic, to obtain their complete physical description, including global properties, one is inevitably lead to incorporate into this picture also gravity.[1] On the other hand, one can envisage these or similar structures within the general relativity (GR) framework only, especially when their field theoretical justification is not available or hard to conceive without more sophisticated models, usually invoking additional, hypothetical processes. The case at hand is the cosmic spinning string [3] which can be viewed as a generalization of a static cosmic string [1, 2]. Unlike the latter, the spinning cosmic string carries angular momentum whose source cannot be convincingly and decidedly established. The only attempt to elucidate the origin of the angular momentum made by Mazur [6] relies on a purely microscopic phenomenon of superconductivity. This, however, is not the only problem that one encounters while studying the objects under discussion. Some of them are described by sources which themselves are challenged by certain pathologies. A good example to call for is that of the global monopole [4]. The energy-momentum tensor of that object is provided by a respectable theory. However, since the mass of the monopole per radial length is constant, its density grows infinitely large when we approach the monopole's center. Due to this and the physical requirement that the monopole's mass be finite one should think of a realistic gravitational monopole as a shell rather than a ball.

The still hypothetical objects mentioned in the previous paragraph belong to some sort of extremes: the spinning cosmic string in the way it is described by [3] extends to infinity, as does the gravitational monopole according to the model of [4]. However, these circumstances should not discourage us from considering those models as representing some physical reality to which they are just the first and sometimes crude approximation. In fact, it is usually very difficult to find metrics that would give us finer descriptions of this reality without the simplicity to be sacrificed, be it a finite cosmic string or a global monopole of a finite size. Therefore such models are not unfrequently the most reasonable compromises between the complexity that a more accurate description necessarily entails and the elucidation of physical contents which one attempts to achieve without pushing the complexity too far. Since it is very often the latter that interests us most, it would be rather unwise to discard the models that are capable of giving us a glimpse of the contents at a very little expense of means involved. As we will see, this also concerns the model we aim at presenting here, the gravitational dipole, which shares features of the global monopole both in its infinite spatial extension and a singularity of its internal physical characteristics, in this case the density of angular momentum. We do propose to overcome these by suggesting some better physical approximation to the strict mathematical model in a manner similar to that for the global monopole.

The nature of spin inducing sources in GR is still a subject of speculations. For example, the Kerr solution [7] that we will use to derive the metric of the gravitational dipole, believed to be the metric of the exterior of the rotating black hole, has not been successfully matched to an interior solution that would be generated by some material model, like a spinning shell. Some progress towards this goal has been achieved [9, 10], though.

Despite this rather obscure nature of the spinning mechanisms in GR, or perhaps because of this,

[1]The history of topological defects in gravity begins with a seminal paper of A. Staruszkiewicz in *Acta Phys. Polon.* **24**, 734 (1963), where the defect analogous to the cosmic string is introduced in the 1+2- dimensional context.

one is nevertheless challenged to try to understand how the spinning objects can arise and to what physical phenomena they may lead in either classical or quantum domain of physics. The goal of the present paper is to address the latter issue. In what follows we propose a model of the 3-dimensional gravitational dipole in an attempt to investigate the consequences of spinning motions in a general relativistic setting. It is in this setting that the gravitational dipole model seems to emphasize the role of spin in a more explicit manner than the Kerr black hole. In the limit of vanishing angular momentum, the Kerr metric boils down to the Schwarzschild metric, both describing curved manifolds, while in the very same limit the gravitational dipole space-time becomes flat. It is in this context that the difference between the two spinning space-times is demonstrated most pronouncedly. The content of the gravitational dipole metric is purely rotational, without the angular momentum any conceivable gravitational interactions are absent here, unlike in the case of the Kerr metric. One can therefore hope that through the study of this model one would better understand the physical implications of spin in GR in three dimensions which may be different than those due to an essentially two dimensional spinning source of the string [5]. One would also like to understand how these sources can affect the behavior of spin carrying quantum particles. These investigations will be presented in another paper [11].

The paper is organized as follows. The next section deals with the derivation of the metric for the model under consideration and discusses the conditions under which this derivation is valid. It also presents the main properties of the model's space-time and considers their physical relevance. As explicitly shown in the appendix, unlike in the case of the spinning string the space of the gravitational dipole is not even locally flat. Because of this, the spin induced gravity affects the motion of test particles in a manner different from the way the spinning cosmic string imparts on them. This motion is more complex than in the field of the string as new features characteristic of the centrifugal force are present. This is the main observation of the section that follows the derivation of the metric and precedes the conclusions that summarize our findings.

2 The metric of the gravitational dipole and assumptions that underline it

We will derive here the metric of the gravitational dipole and discuss the circumstances under which it can be valid as a consistent description of some sort of physical reality. To this end, we start from the Kerr metric in the Boyer-Lindquist coordinates [8]

$$\begin{aligned} ds^2 = {} & \tfrac{\Delta}{\Sigma^2}(dt - a\sin^2\theta d\phi)^2 - \tfrac{\sin^2\theta}{\Sigma^2}[(r^2+a^2)d\phi - ad\phi]^2 - \tfrac{\Sigma^2}{\Delta} - \Sigma^2 d\theta^2 = \\ & \tfrac{1}{\Sigma^2}(\Delta - a^2\sin^2\theta)dt^2 + \tfrac{2\sin^2\theta}{\Sigma^2}[a(r^2+a^2) - a\Delta]dtd\phi - \tfrac{\Sigma^2}{\Delta}dr^2 - \Sigma^2 d\theta^2 + \\ & \tfrac{\sin^2\theta}{\Sigma^2}[a^2\Delta\sin^2\theta - (r^2+a^2)^2]d\phi^2, \end{aligned} \tag{1}$$

where $\Sigma^2 = r^2 + a^2\cos^2\theta$, $\Delta = r^2 - 2Mkr + a^2$, and $a = \frac{J}{M}$. Here J denotes the angular momentum of the rotating black hole, M its mass, and $k = \frac{G}{c^2}$. We will assume that $J = jr$, $M = mr$, j and m being some constants and that terms proportional to and of higher order in $p = \frac{a}{r}$ can be neglected in a certain sensible approximation that we want to work out. This amounts to saying that $p \ll 1$ in the approximation intended. We will later substantiate this approach in greater detail. We will also assume that $km \ll 1$, $kj \ll 1$ and because of this the terms kmp (and obviously also kmp^2) will not be taken into account in a series expansion of the metric (1) in parameters km, kj, and p that seem to be the most natural ones for this sort of expansion.

Now, from (1) we obtain up to $O(p^2)$ but with $O(kmp^2)$ already discarded that

$$\begin{array}{lll} g_{tt} = 1 - 2km, & -g_{rr} = 1 + 2km - p^2 \sin^2\theta, & -g_{\theta\theta} = r^2(1 + p^2\cos^2\theta), \\ -g_{\phi\phi} = r^2(1+p^2)\sin^2\theta, & -g_{t\phi} = 2kj\sin^2\theta. & \end{array} \tag{2}$$

This leads us to the metric:

$$\begin{array}{r} ds^2 = (1-2km)(dt - 2kj\sin^2\theta d\phi)^2 - (1+2km)(1 - \frac{p^2\sin^2\theta}{1+2km})dr^2 - \\ r^2[(1+p^2\cos^2\theta)d\theta^2 + (1+p^2)\sin^2\theta d\phi^2] + O(k^2j^2)d\phi^2. \end{array} \tag{3}$$

Neglecting terms $O(k^2j^2)$ and $O(p^2)$ gives

$$ds^2 = (1-2km)(dt - 2kj\sin^2\theta d\phi)^2 - (1+2km)dr^2 - r^2 d\theta^2 - r^2\sin^\theta d\phi^2. \tag{4}$$

Rescaling (4) by $(1-2km)^{-1}$ and changing the coordinates via $d\rho = (1+2km)dr$ brings us to

$$ds^2 = (dt - 2kj\sin^2\theta d\phi)^2 - d\rho^2 - (1-2km)\rho^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{5}$$

where we used $(1-2km)^{-1} = 1 + 2km$ to be consistent with our assumption $km \ll 1$. Therefore we arrived at the metric

$$ds^2 = (dt - A\sin^2\theta d\phi)^2 - d\rho^2 - B\rho^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{6}$$

where $A = 2kj$ and $B = 1 - 2km$. Since $B \neq 1$ represents a purely monopole contribution to the metric (6) as seen from the metric of the global monopole [4],

$$ds^2 = dt^2 - d\rho^2 - b^2\rho^2(d\theta^2 + \sin^2\theta d\phi^2),$$

and we are interested in the purely dipole part of it, we will put B equal 1 in order to focus on the dipole content of our derivation. This is tantamount to saying that

$$ds^2 = (dt - A\sin^2\theta d\phi)^2 - dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{7}$$

is the metric of the gravitational dipole under the assumptions specified earlier.

We will now investigate if these assumptions are self-consistent and what is the physical nature of the object that produces the gravitational field described by (7). The source of the field is the momentum-energy tensor whose components are proportional to the components of the Einstein tensor and the latter are worked out in the appendix. Because of their complexity it is rather hard to propose a field-theoretical model, or what amounts to the same, some matter Lagrangian that would lead to this energy-momentum tensor. This, however, as argued in the introduction, should not be considered a major obstruction. The metric (7), due to the assumption that the angular momentum per unit radial length is constant, describes an object whose total angular momentum is infinite. To make this situation look more physical one can think of (7) as a good approximation to the metric of an exterior of a shell of thickness d in its vicinity, a distance $r \gg a$ from the center, which is required to satisfy the condition $p \ll 1$. Then the angular momentum is finite and proportional to d. We will soon show that under reasonable and physically natural circumstances one can make $a = 1$ and since making $a = 1$ coincides with imposing on r the condition $r \geq \frac{1}{(km)^{1/2}}$ the discussed condition $p \ll 1$ is not restrictive at all. As a matter of fact, because $km \ll 1$, $p \ll 1$ is easily met. Why it should

be a shell rather than a ball can be justified by equilibrium considerations. For the equilibrium to take place the centripetal force present in any rotating system should not overcome the gravity force, that is for a test particle of mass μ there should be $\mu\omega^2(r)r \leq \frac{kM(r)\mu}{r^2}$, where $\omega(r)$ and $M(r)$ are the angular velocity and the mass of the matter that generate our metric. This means that

$$\omega^2(r) \leq \frac{kM}{r^3} = kmr^2. \tag{8}$$

On the other hand, from the definition of the angular momentum $J = \int r^2\omega(r)\, dM(r)$ one obtains

$$\frac{dJ}{dr} = j = const = r^2\omega(r)\frac{dM}{dr} = r^2\omega(r)m \tag{9}$$

according to the assumptions employed. Eq. (9) requires $\omega(r) \sim \frac{1}{r^2}$, thus leading to a sigularity with $r \to 0$. It is the main reason why we want a shell as our model for the metric of the gravitational dipole. The situation here is somewhat similar to the mass distribution in the global monopole, where $\frac{dM}{dr}$ being constant implies $\rho(r) \sim \frac{1}{r^2}$, which holds also for the model of the gravitational dipole under study. Obviously, ω cannot grow to infinity without destroying the stability of the whole configuration. It is Eq. (8) that demands for $\omega(r) = \frac{C}{r^2}$ a minimum radius r_0 such that $\frac{C^2}{r_0^4} = \frac{km}{r_0^2}$ or $r_0 = \sqrt{\frac{C^2}{km}}$. By choosing $C = 1$, which can always be accomplished by rescaling the length, we see that the balance of the configuration is maintained as long as its inner radius is greater than $\frac{1}{(km)^{1/2}}$. This choice of C leads via (9) to $a = \frac{j}{m} = 1$.

A few remarks on physical properties of the introduced space-time seems to be in order here. First of all, the metric (7) as an axially-symmetric one possesses two Killing vectors ∂_t and ∂_ϕ. In the region $r < A\sin\theta$, the latter, which has closed orbits, becomes timelike. Any observer following these orbits would experience causality violation. Fortunately, in the regime in which the metric of the gravitational dipole is valid (that is $km \ll 1$ and so also $kj \ll 1$ for $a = \frac{j}{m} = 1$) this region lies inside the shell discussed above and its physical relevance is only apparent. As long as one stays outside of this shell, one can dismiss this issue.

As shown in [12] the spinning cosmic string indroduces the time delay between the arrival time of two particles moving in opposite directions that follow an arbitrary closed trajectory around the string. One can rightly expect a similar effect in the case under study. The only difference between these two physical situations is that in the metric of the gravitational dipole this delay depends on the angle θ. For light rays describing circular paths with $\phi = \Omega t$, $r = R$ =const, and θ =const, the eikonal equation $ds^2 = 0$ implies $\Omega_+ = \frac{1}{R - A\sin^2\theta}$ and $\Omega_- = -\frac{1}{R + A\sin^2\theta}$ with the Ω_+ assigned to the ray moving along with the direction of rotation. Assuming $R > A\sin^2\theta$, which, as argued earlier, is always valid for our model, one obtains for the time delay

$$T = 2\pi\left[\frac{1}{|\Omega_-|} - \frac{1}{\Omega_+}\right] \tag{10}$$

The last comment concerns the very name we chose for the model presented. As is well-known, the gravitational charges are only alike, thus preventing the existence of "electric" gravitational dipoles. However, the dipole we introduced in this paper is of "magnetic" nature. Since in GR every form of energy contributes to the gravitational field, in the case under consideration it is the rotational ("magnetic") energy of the dipole that does so. As opposed to the spinning cosmic string that carries

a nonzero flux of the “magnetic” field, but does not produce a detectable field itself [2], the gravitational dipole creates such a field as manifested by the curvature of its space-time.

3 The motion of test particles

As observed in the previous section, the metric of the gravitational dipole possesses two Killing vectors, ∂_t and ∂_ϕ that can be used to find integrals of motion of test particles in the space-time of the dipole. The integrals themselves are useful in simplifying the equations of motion for the particles by casting them into a dynamical system. To employ this strategy let us recall that

$$(\partial_\alpha|\dot{\xi}) = \text{const} = \frac{1}{\mu_0}P_\alpha \tag{1}$$

along the geodesic ξ, where ∂_α is a Killing vector, $\dot{\xi}$ is a vector tangent to the geodesic, and P_α is a momentum conjugate to the coordinate x^α, chosen so that ∂_α is tangent to it. Eq. (1) is valid also for a zero-mass particles, i.e., when $\mu_0 = 0$ except for its last equality. The brackets in (1) denote the scalar product of two vectors separated by a vertical dash.

Eq. (1) when applied to an axially-symmetric metric yields

$$g_{tt}\dot{t} + g_{t\phi}\dot{\phi} = -\frac{E}{\mu_0 c^2} = -\gamma, \tag{2a}$$

$$g_{tt}\dot{t} + g_{\phi\phi}\dot{\phi} = \frac{l}{\mu_0 c} = \lambda, \tag{2b}$$

where γ and λ are constants related to the energy E and the azimuthal angular momentum l of the test particle. When used for our metric Eqs. (2) lead to

$$\dot{t} = \frac{\lambda A - \gamma X}{r^2}, \tag{3a}$$

$$\dot{\phi} = \frac{\lambda - A\gamma}{r^2}, \tag{3b}$$

where $X = r^2 - A^2\sin^2\theta$. The remaining equations of motion can be obtained from the Hamilton-Jacobi equation

$$g^{\mu\nu}\frac{\partial S}{\partial x^\mu}\frac{\partial S}{\partial x^\nu} - \mu_0^2 c^2 = 0. \tag{4}$$

In this formalism $P_\mu = \frac{\partial S}{\partial x_\mu}$ and since we have already established that $P_t = -\mu_0 c\gamma$ and $P_\phi = \mu_0 c\lambda$ one easily arrives at

$$\left(1 - \frac{A^2\sin^2\theta}{r^2}\right)P_t^2 - \frac{1}{r^2}P_\theta^2 - P_r^2 - \frac{2A}{r^2}P_tP_\phi - \frac{1}{r^2 sin^\theta}P_\phi^2 - \mu_0^2c^2 = 0$$

and then at

$$-\left(\frac{P_\theta^2}{\mu_0^2c^2} + \frac{\lambda^2}{\sin^2\theta} + \gamma^2A^2\sin^2\theta\right) + r^2\left[-\frac{P_r^2}{\mu_0^2c^2} + (\gamma^2 - 1) + \frac{2A\lambda\gamma}{r^2}\right] = 0. \tag{6}$$

[2]For this reason the spinning cosmic string could also be called the gravitational fluxon.

By introducing a separation constant K such that

$$\frac{P_\theta^2}{\mu_0^2 c^2} + \frac{\lambda^2}{\sin^2\theta} + \gamma^2 A^2 \sin^2\theta = K^2 \tag{7}$$

one obtains

$$\frac{P_r^2}{\mu_0^2 c^2} = (\gamma^2 - 1) + \frac{K^2 - 2A\lambda\gamma}{r^2}. \tag{8}$$

With the substitution $\dot{x}^\alpha = \frac{P_\alpha}{\mu_0 c}$ Eqs. (7) and (8) can be reduced to the following dynamical system for r and θ

$$\dot{r} = \left(\gamma^2 - 1 - \frac{K^2 - 2A\lambda\gamma}{r^2}\right)^{1/2}, \tag{9a}$$

$$\dot{\theta} = \left(K^2 - \frac{\lambda^2}{\sin^2\theta} - \gamma^2 A^2 \sin^2\theta\right)^{1/2}. \tag{9b}$$

Making the substitution $z = \cos\theta$, this last equation can be transformed into the energy conservation equation for the one-dimensional motion of unit mass particle in the potential $V(z)$, or explicitly

$$\frac{\dot{z}^2}{2} + V(z) = E \tag{10}$$

with $E = \frac{1}{2}(K^2 - \lambda^2 - \gamma^2 A^2)$ and $V(z) = az^2 + bz^4$, where $a = \frac{1}{2}(K^2 - 2\gamma^2 A^2)$ and $b = \frac{1}{2}\gamma^2 A^2$. One more substitution, $x = z^2$, brings us to the equation

$$\dot{x}^2 = 8x(E - ax - bx^2) \tag{11}$$

which when supplemented with the constraints $\dot{x}^2 \geq 0$ and $x \geq 0$ describes the physical motion in the coordinate θ.

Depending on the values of E and a, there are different ranges of θ in which this motion takes place. Let us first consider the case of $a \geq 0$. As can be easily seen, the motion is possible only for $E \geq 0$, $E = 0$ representing a singular situation with the orbit constrained to the equatorial plane $\theta = \frac{\pi}{2}$. For $E < 0$ both x_- and x_+ are negative and upon consulting with (12) one sees that the motion is impossible ($\dot{x}^2 < 0$). When the "total energy" E is positive, the trajectory is bounded between this plane and the plane determined by the equation $\theta_- = \arccos\sqrt{x_-}$, where $x_- = -\frac{a - \sqrt{a^2 + 4bE}}{2b}$. For $a < 0$ there are more options and only if $E < E_{min} = -\frac{a^2}{4b}$ no motion takes place. For $E = E_{min}$ there exists a stationary orbit in the plane $\theta = \arccos\sqrt{x_2}$ with $x_2 = -\frac{a}{2b}$. This corresponds to the previous situation of $E = 0$. When the "total energy" exceeds E_{min} there exists an orbit bounded by the planes $\theta_- = \arccos\sqrt{x_-}$, and $\theta_+ = \arccos\sqrt{\max[0, x_+]}$, where $x_+ = -\frac{a + \sqrt{a^2 + 4bE}}{2b}$. This also includes the case $E = 0$ for which $\theta_+ = \frac{\pi}{2}$.

If both a and E are negative the solution to (11) can be expressed in terms of elliptic functions. To see this let us rewrite this equation as

$$\dot{x}^2 = -8bx(x - x_+)(x - x_-), \tag{12}$$

where $x_\pm$ are defined above. Upon the introduction of $k^2 = \frac{x_+}{x_-}$ and y such that $x = x_+ y^2$, the last equation becomes

$$\dot{y}^2 = -2bx_-(1 - y^2)(1 - k^2 y^2), \tag{13}$$

Through the substitution $u = i\sqrt{2bx_+}t$ one can bring (13) to the standard form of the differential equation for elliptic functions

$$\left(\frac{dy}{du}\right)^2 = (1-y^2)(1-k^2y^2),$$

whose solution is $y(u) = sn(u+\delta)$ with δ being some constant. Finally, the solution to (9b) is found to be

$$\theta(t) = \arccos\left[\sqrt{x_+}\left|\mathrm{sn}\left(\sqrt{2bx_-}it + \delta\right)\right|\right]. \tag{14a}$$

If $E = 0$ and $a < 0$ then $x_+ = 0$ and (12) reduces to

$$\left(\frac{dy}{du}\right)^2 = 2bx_+y^2(1-y^2),$$

where $y^2 = \frac{x}{x_-}$. Upon the substitutions $u = \sqrt{2bx_-}t$ and $1 - y^2 = w^2$ the last equation boils down to

$$\left(\frac{dw}{du}\right)^2 = (1-w^2)^2,$$

solution to which is given by $w = \pm\tanh(u+c)$, c being some constant. Eventually, we find that

$$\theta(t) = \arccos\left[\sqrt{x_-}\left|1 - \tanh^2\left(c + \sqrt{2bx_-}t\right)\right|\right]. \tag{14b}$$

For E positive and independently of a, the trajectory is bounded between the equatorial plane and the plane $\theta_- = \arccos\left(-\frac{a-\sqrt{a^2+4bE}}{2b}\right)^{1/2}$. This case can be reduced to the previous one by changing x_+, which is always negative, to $-|x_+|$, and replacing x by y via $x = y - |x_+|$. In doing so one ends up with

$$\dot{y}^2 = -8by\left(y - (x_- + |x_+|)\right)$$

that looks exactly like the equation leading to (14b), except that x_- gets replaced by $x_- + |x_+|$. Obviously, the solution to (9b) for such conditions is

$$\theta(t) = \arccos\sqrt{(x_- + |x_+|)\left(1 - \tanh^2\left(c + \sqrt{2b(x_- + |x_+|)}t\right)\right)^2 - |x_+|}. \tag{14c}$$

However, the fact that the motion is bounded within some range of θ does not necessarily mean that its orbits are closed. As a matter of fact, we will now demonstrate that the orbits of test particles are open. To this end let us find the trajectory of the test particle as a function $r(\phi)$. By combining (3b) and (9a) one obtains

$$d\phi = \frac{Cdr}{r^2(\alpha^2 - \beta^2/r^2)}, \tag{15}$$

where $C = \lambda - \gamma A$, $\alpha^2 = \gamma^2 - 1 > 0$, and $\beta^2 = K^2 - 2A\lambda\gamma$. Let us now consider three cases of $\beta^2 > 0$, $\beta^2 = 0$, and $\beta^2 < 0$. They will lead to different classes of trajectories. In the first case, changing variables $\frac{1}{r} = u$ subsequently $u = \frac{\beta}{\alpha}\cos\chi$ brings one to the solution

$$\frac{1}{r(\phi)} = \frac{\alpha}{|\beta|}\cos\left[|\beta|(\phi + B)/C\right], \tag{16}$$

where B is a constant of integration. The case $\beta^2 = 0$ is the easiest to work out. Through only one change of variables $\frac{1}{r} = u$, it immediately leads to

$$r(\phi) = -\frac{C}{\alpha(\phi + B)}. \tag{17}$$

In the last case, the change $u = \frac{\alpha}{|\beta|}\sinh\chi$ enables one to arrive upon integration at

$$\frac{1}{r(\phi)} = \frac{\alpha}{|\beta|}\sinh\left[-|\beta|(\phi + B)/C\right]. \tag{18}$$

As seen from (3b), $C > 0$ ensues that ϕ increases with time. However, the sign of C has no bearing on the character of motion, all it does is to change its direction. The trajectories in the last two cases describe unfolding spirals which asymptotically approach infinity in the limit $\phi \to B$. For $C > 0$ this is achieved through increasing values of ϕ that can run from $-\infty$ while for $C < 0$ the orbit develops towards decreasing values of ϕ with ϕ running from $+\infty$. That it is so is determined by the condition $r(\phi) \geq 0$. In the first case the trajectories are hyperbolas unless $\beta = \pm C$ which corresponds to $E = 0$ and results in a motion along a straight line in the equatorial plane. Both the hyperbolas and the straight line make the closest approach to the center a distance $\frac{|\beta|}{\alpha}$ from it. The asymptotes of the hyperbolas are defined by the equation $|\beta|(\phi + B)/C = \pm\frac{\pi}{2}$. When the motion is projected onto the equatorial plane, the asymptotes divide this plane into four sectors with the particle moving in the one that satisfies the condition $\cos\left[|\beta|(\phi + B)/C\right] > 0$.

The equations of motion for particles of zero rest mass can be worked out in a similar manner. In fact, if the mass term is dropped in (5) the only change it brings about is in Eqs. (6), (8), and (9a), where $\gamma^2 - 1$ is replaced by γ^2 which represents the square of the total energy of the particle. The way this change affects the presented solutions is really insignificant.

4 Conclusions

We presented the model of the gravitational dipole in a yet another attempt to investigate the impact of spinning sources on gravity in a general relativistic framework. The model studied has some advantages in this respect over the Kerr black hole as its gravitational field is of pure rotational origin, thus allowing one to examine this problem in its disentanglement from the main gravity source, the mass. We showed that the model has sensible properties if it is thought of as an approximation to a rotating shell. The space-time of the dipole is curved, therefore it should not come as a surprise that the way gravity associated with the curvature affects the motion of test particles is more pronounced and complex than that of the spinning cosmic string. Unlike for the spinning defect-free string, the trajectories of test particles in the field of the dipole when projected on the equatorial plane are not straight lines anymore, the only exception being the motion in the plane itself. All trajectories are open and bounded in the θ coordinate.

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Appendix

The only non-zero covariant components of the Einstein tensor for the metric of the gravitational dipole are as follows

$$G_{tt} = -\frac{3a^2\cos^2\theta}{r^4},$$

$$G_{rr} = \frac{a^2\cos^2\theta}{r^4},$$

$$G_{\theta\theta} = \frac{a^2\cos^2\theta}{r^2},$$

$$G_{\phi\phi} = -\frac{a^2(5r^2\cos^2\theta - 3r^2\cos^4\theta + 3a^2\cos^2\theta - 6a^2\cos^4\theta + 3a^2\cos^6\theta)}{r^4},$$

$$G_{\phi t} = G_{t\phi} = -\frac{a(r^2 - r^2\cos^2 - 3a^2\cos^2\theta + 3a^2\cos^4\theta)}{r^4}.$$

The Ricci scalar R is

$$R = \frac{2a^2\cos^2\theta}{r^4}.$$

Clearly, the space-time of the gravitational dipole is curved.

References

[1] T. W. B. Kibble, *J. Phys. A* **9**, 1387 (1976).

[2] A. Vilenkin, *Phys. Rev. D* **23**, 852, (1981).

[3] P. O. Mazur, *Phys. Rev. Lett.* **57**, 929 (1986); **59**, 2379 (1987).

[4] M. Barriola and A. Vilenkin, *Phys. Rev. Lett.* **63**, 341 (1989).

[5] S. Deser, R. Jackiw, and G. 't Hooft, *Ann. Phys.* (N. Y.) **152**, 220 (1984).

[6] P. O. Mazur, *Phys. Rev. D* **34**, 1925 (1986).

[7] R. P. Kerr, *Phys. Rev. Lett.* **11**, 237 (1963).

[8] R. H. Boyer and R. W. Lindquist, *J. Math. Phys.* **8**, 265 (1966).

[9] V. De La Cruz and W. Israel, *Phys. Rev.* **170**, 1187 (1968).

[10] W. Israel, *Phys. Rev. D* **2**, 641 (1970).

[11] W. Puszkarz, in progress.

[12] D. Harari and A. P. Polychronakos, *Phys. Rev. D* **38**, 3320 (1988).